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Recursion operators and constants of motion in supermechanics

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Abstract: We prove that only even graded Poisson brackets can be characterized by the vanishing of the graded Schouten bracket of the associated graded tensor of type $(0, 2)$ with itself. On the other hand, we prove that the supertraces of different powers of an invariant graded tensor of type $(1, 1)$ are constants of motion, and that when such tensor is even, the infinitesimal supersymmetries it generates out of a given infinitesimal supersymmetry form a supercommutative superalgebra.

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1. Introduction

The application of advanced methods and tools of Differential Geometry in the study of physical problems during the last two decades has thrown a great deal of light in many problems, and in particular, in the study of dynamical systems, where the use of symplectic and Poisson structures has been shown to be very fruitful. So, constants of motion are obtained using different versions of Noether's theorem or by alternative methods. For instance, it is by now well known that the knowledge of a Γ-invariant tensor field T of type $(1, 1)$ provides constants of motion for the dynamics described by the vector field Γ: the traces of different powers of T, and moreover, such a tensor T may be considered as generating symmetry, because when applied to an infinitesimal symmetry X leads to a new such symmetry $T(X)$. The conditions for such constants of motion being independent and in involution, and then leading to completely integrable systems, both in the Lagrangian and Hamiltonian approach, can be examined using the Nijenhuis tensor N_T of T. The possibility of extending this formalism in order to include fermionic dynamical systems was analysed in [17,10,18], where it was told that only even recursion operators are interesting, otherwise the Nijenhuis tensor is meaningless. This was exemplified with the mixed bosonic-fermionic harmonic oscillator, the Toda lattice and the Witten dynamics.

The aim of this paper is to review and discuss the generalization of these matters in the graded context. In particular, we want to analyse to what extent this generalization depends

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on the parity of the geometric structure involved. Thus, we shall see that many properties and results of the classical case have their counterpart in the graded case when the structure at hand is even, whereas in the odd case some difficulties arise, and we want to investigate the reason for this to be so.

The paper is divided in two sections. In the first one we study the arena where the Hamiltonian formalism of supermechanics is developed, namely, graded symplectic and Poisson manifolds. We show that when the geometric structure is even the underlying manifold has associated, in a natural and canonical way, a classical structure, which explains the similarity of the classic and the even graded geometry. Also in this section, we study the Schouten bracket in the graded context; we prove that when the graded Poisson structure is even it can be described, in analogy with the classical case, in terms of this bracket. Surprisingly enough, we conclude the section noting that the Schouten bracket cannot be defined when the graded Poisson structure is odd.

In the second section we review the definition of the super-Nijenhuis tensor of a graded tensor of type $(1, 1)$, which is based on a generalization of the relation $d_{N_T} = [d_T, d_T]$, valid for $(1, 1)$-tensor fields. Finally, the definition of recursion operator or generating symmetry as a Γ-invariant tensor field of type $(1, 1)$ is given, and many of the properties of such tensor are analysed in the supercase. So it will be shown that the supertraces of the different powers of the graded tensor T are constants of motion and we shall study more carefully the particular case in which the graded Nijenhuis tensor vanishes.

2. Symplectic and Poisson supermanifolds

2.1. Symplectic supermanifolds

One of the central points that we want to stress in this paper is the difference between even and odd geometries. We start this comparison by reviewing the notion of a symplectic supermanifold. Although the difference at this level is not so clear cut, or at least does not look so striking, a different type of mechanics is obtained when one considers an odd symplectic structure, and new interesting Lie superalgebras that have no analogue in the non-graded geometry do appear [20, 12].

Throughout this paper we shall be dealing with the so called graded manifolds, in other words, we shall adopt the definition of supermanifold given in the pioneer works of Kostant [15] and Berezin–Leĭtes [2, 19]. Nevertheless, we shall use the notation as in [5, 6]. In particular, a (m, n)-dimensional graded manifold will be denoted by $\mathcal{M} = (M, \mathcal{A})$, the locally free sheaf of $\mathcal{A}_{\mathcal{M}}$-modules $\mathrm{Der}\,\mathcal{A}$, which corresponds to the graded vector fields, will be denoted by $\mathfrak{X}(\mathcal{A})$ and the dual sheaf, the sheaf of graded 1-forms, will be denoted by $\Omega^1(\mathcal{A})$. In general, graded k-forms are defined by $\Omega^k(\mathcal{A}) := \bigwedge^k(\Omega^1(\mathcal{A}))$, where the wedge product is to be understood in the sense of graded algebras; moreover the sheaves of graded tensors are defined using the standard graded tensor algebra [15, 1] on the sheaf of $\mathcal{A}$-modules $\mathfrak{X}(\mathcal{A})$. Thus, graded tensor fields of type (k, l) will be regarded as graded multilinear forms:

$$\mathcal{T}_l^k(\mathcal{A}) = \mathcal{B}\Big(\underbrace{\mathfrak{X}(\mathcal{A})^*, \ldots, \mathfrak{X}(\mathcal{A})^*}_{l}, \underbrace{\mathfrak{X}(\mathcal{A}), \ldots, \mathfrak{X}(\mathcal{A})}_{k}; \mathcal{A}\Big).$$

If ω is a homogeneous graded 2-form on $\mathcal{M}$, then one has a morphism of graded $\mathcal{A}$-modules $\flat\colon \mathfrak{X}(\mathcal{A}) \to \Omega^1(\mathcal{A})$, defined by $X^\flat := i_X\omega$ for $X \in \mathfrak{X}(\mathcal{A})$, which satisfies $|\flat| = |\omega|$. As in the non-graded case, we will say that ω is *non-degenerate* when $\flat$ is an isomorphism, and a *graded symplectic manifold* is a pair $(\mathcal{M}, \omega)$, consisting of a graded manifold $\mathcal{M}$ and a closed, non-degenerate, homogeneous graded 2-form ω on $\mathcal{M}$.

If $\mathcal{U}$ is an open subset of M such that $(\mathcal{U}, \mathcal{A}(U))$ is isomorphic to a superdomain, then the restriction of ω to $\mathcal{U}$ is associated to a graded matrix whose elements are superfunctions in $\mathcal{A}(U)$, and the non-degeneracy condition reduces to the fact that this matrix is non-singular. Moreover, from this matrix one can get information about the dimension of $\mathcal{M}$, and since here is the first place where parity enters in the game we shall provide the details in the following:

Proposition 2.1. *Let $(\mathcal{M}, \omega)$ be a graded symplectic manifold of dimension (m, n). If $|\omega| = 0$, then $(M, R^*\omega)$, where R denotes the augmentation map, is a symplectic manifold. In particular, m is an even number.*

Proof. Since $R\colon \big(M, C^\infty(M)\big) \to (M, \mathcal{A})$ is a morphism of graded manifolds, $R^*\omega$ is a closed graded 2-form (for an intrinsic definition of the pull back of a graded form see [6]).

On the other hand, let $\mathcal{U}$ be an open subset of M such that $(\mathcal{U}, \mathcal{A}(U))$ is isomorphic to a superdomain, let $\{ q^i, \theta^\alpha \}$ be a supercoordinate system of $\mathcal{M}$ on $\mathcal{U}$, and assume that the matrix associated to the restriction of ω to $\mathcal{U}$ in those supercoordinates is

$$\omega = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{2.1}$$

Thus, if $\tilde{\omega}$ denotes the matrix whose entries are the image of the entries of ω under the morphism of superalgebras $r^*\colon \mathcal{A}(U) \to C^\infty(\mathcal{U})$, then ω is invertible if, and only if, $\tilde{\omega}$ is invertible [19]. Moreover, since $|\omega| = 0$

$$\tilde{\omega} = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{D} \end{pmatrix},$$

and $\tilde{\omega}$ is invertible exactly when both $\tilde{A}$ and $\tilde{D}$ are invertible. To conclude, we notice that the matrix associated to the restriction of $R^*\omega$ to $\mathcal{U}$, in the coordinate system $\{ \tilde{q}^i \}$, is precisely $\tilde{A}$. $\square$

From this proposition it is natural to expect that the even symplectic geometry will be quite similar to the standard symplectic geometry, since, in a way, the former is an extension of the latter. A detailed description of even symplectic graded manifolds in terms of objects of the usual differential geometry can be found in [22].

On the other hand, when $|\omega| = 1$, it turns out that

$$\tilde{\omega} = \begin{pmatrix} 0 & \tilde{C} \\ \tilde{D} & 0 \end{pmatrix},$$

and since $\tilde{C}$ is a $m \times n$ matrix, then ω will be non-degenerate only when $m = n$. In this case the geometry is not so similar to the standard symplectic geometry. Moreover, even the generalization of Darboux theorem takes a different look depending on whether the graded symplectic form is even or odd, see [20, 23] (the even case was already discussed in [15]).

2.2. Graded Poisson manifolds

If $(\mathcal{M}, \omega)$ is a symplectic graded manifold and μ is a graded 1-form, we denote its inverse image under the isomorphism $\flat\colon \mathfrak{X}(\mathcal{A}) \to \Omega^1(\mathcal{A})$ by $X_\mu \in \mathfrak{X}(\mathcal{A})$. Moreover, since $\mathfrak{X}(\mathcal{A})$ is a Lie superalgebra we can translate this structure to $\Omega^1(\mathcal{A})$ via the *graded Poisson bracket* by $\{\mu, \nu\} := i_{[X_\mu, X_\nu]}\omega$, which has the following properties:

(i) $\{\cdot\,,\cdot\}$ is $\mathbb{R}$-bilinear,

(ii) $\{\mu, \nu\} = -(-1)^{(|\omega|+|\mu|)(|\omega|+|\nu|)}\{\nu, \mu\}$,

(iii) $(-1)^{(|\omega|+|\mu|)(|\omega|+|\rho|)}\{\mu, \{\nu, \rho\}\} + \cdots = 0$,

were $\cdots$ stands for the terms arising from cyclic permutations of μ, ν and ρ. In particular, if $|\omega| = 0$, then $(\Omega^1(\mathcal{A}), \{\cdot\,,\cdot\})$ would also be a sheaf of Lie superalgebras.

Now, every superfunction f has associated a graded 1-form, its differential df, and therefore it also has associated a supervector field, to wit $X_f := X_{df}$. Moreover, since the graded Poisson bracket of two exact graded 1-forms is again an exact graded 1-form:

$$\{df, dg\} = -(-1)^{|\omega|(|X_f|+|X_g|)}d\big(\omega(X_f, X_g)\big),$$

then one can transport the graded Poisson bracket to the level of superfunctions via the formula:

$$\{f, g\} := -(-1)^{|\omega|(|X_f|+|X_g|)}\omega(X_f, X_g),$$

when $f, g \in \mathcal{A}$ are homogeneous superfunctions. One can check that this bracket has similar properties to its counterpart in classical mechanics, in particular, it can be defined alternatively by $\{f, g\} = X_f(g) = -(-1)^{|X_f||X_g|}X_g(f)$.

Naturally one would like to define a graded Poisson manifold as a graded manifold such that the sheaf $\mathcal{A}$ is actually a sheaf of graded Poisson algebras; unfortunately only when $|\omega| = 0$, $(\mathcal{A}, \{\cdot\,,\cdot\})$ is a graded Poisson algebra. In the odd case one gets instead what is known in the literature as a Gerstenhaber algebra. Thus, the concept of graded Poisson manifold needs a slight modification if we want every graded symplectic manifold to be a graded Poisson manifold in a natural way.

Definition 2.1. A *graded Poisson manifold of parity p* is a graded manifold, $(M, \mathcal{A})$, such that in the superalgebras $\mathcal{A}(U)$, of the structural sheaf, there is an extra product $\{\cdot\,,\cdot\}_{\mathcal{U}}$, compatible with the restrictions, and such that for $f, g, h \in \mathcal{A}(U)$ the following properties hold:

(a) $\{f, g\}_{\mathcal{U}} = -(-1)^{(|f|+p)(|g|+p)}\{g, f\}_{\mathcal{U}}$,

(b) $(-1)^{(p+|f|)(p+|h|)}\{f, \{g, h\}\} + \cdots = 0$,

(c) $\{f, gh\}_{\mathcal{U}} = \{f, g\}_{\mathcal{U}}h + (1)^{|g|(p+|f|)}g\{f, h\}_{\mathcal{U}}$.

Once again, $\cdots$ stand for the terms arising from cyclic permutations.

The reader particularly interested in graded Poisson brackets can find more details in references [3, 4, 16, 11].

2.3. Schouten bracket

As in the non-graded case, any graded Poisson structure $\{\cdot\,,\cdot\}$ on $\mathcal{M}$, has associated a graded tensor of type $(2,0)$, which is defined by the equation

$$\Delta(df, dg) := \{f, g\} \qquad (\forall f, g \in \mathcal{A}).$$

Moreover, if the Poisson structure is of parity p, then

$$\Delta(df, dg) = -(-1)^{(|f|+p)(|g|+p)}\,\Delta(dg, df). \tag{2.2}$$

In particular, when $p = 0$, Δ is antisymmetric. On the other hand, if $p = 1$, then Δ is neither symmetric nor antisymmetric; this is a warning that some complications might arise in this case. So let us first concentrate on even Poisson structures.

Let $\mathcal{U}$ be an open subset of M such that $(\mathcal{U}, \mathcal{A}(U))$ is isomorphic to a superdomain. If $\{q^i, \theta^\alpha\}$ are local supercoordinates of $\mathcal{M}$ on $\mathcal{U}$, then Δ can locally be described by the superfunctions

$$\Delta^{ij} := \Delta(dq^i, dq^j) = \{q^i, q^j\}, \qquad M^{i\beta} := \Delta(dq^i, d\theta^\beta) = \{q^i, \theta^\beta\},$$
$$M^{\alpha j} := \Delta(d\theta^\alpha, dq^j) = \{\theta^\alpha, q^j\}, \qquad S^{\alpha\beta} := \Delta(d\theta^\alpha, d\theta^\beta) = \{\theta^\alpha, \theta^\beta\};$$

and the antisymmetry of Δ implies that

$$\Delta^{ij} = -\Delta^{ji}, \qquad M^{i\alpha} = -M^{\alpha i}, \qquad S^{\alpha\beta} = S^{\beta\alpha}. \tag{2.3}$$

On the other hand, to each superfunction $f \in \mathcal{A}$, we can associate the supervector field X_f, defined by $X_f(g) = \{f, g\}$. In particular, $X_{q^i}(q^j) = \Delta^{ij}$, and so on. Moreover, a careful bookkeeping, using the antisymmetric properties (2.3), gives

$$\Delta|_{\mathcal{U}} = \sum_{k=1}^{m} \partial_{q^k} \wedge X_{q^k} - \sum_{\gamma=1}^{n} \partial_{\theta^\gamma} \wedge X_{\theta^\gamma}. \tag{2.4}$$

Now, there are several alternatives to define the Schouten bracket in non-graded geometry (see, for instance, [21, 7]); all of them can be generalized to the graded context, but we shall follow what Roger [21] called the algebraic approach.

Let $\Lambda(\mathcal{A})$ be the exterior algebra, in the graded sense, of the Lie superalgebra $\mathfrak{X}(\mathcal{A})$, so $\Lambda(\mathcal{A})$ is the set of all antisymmetric graded tensor of type $(k, 0)$. The Schouten bracket is defined inductively using a derivation property, keeping in mind that the algebra at hand is a $\mathbb{Z} \oplus \mathbb{Z}_2$ graded algebra.

Definition 2.2. If X, Y and $Z \in \Lambda(\mathcal{A})$ are homogeneous graded tensors of type $(0, k)$, $(0, l)$ and $(0, r)$ respectively, the *Schouten bracket* is the antisymmetric bilinear form $[\cdot,\cdot]_S$ that satisfy the following derivation property

$$[X, Y \wedge Z]_S = [X, Y]_S \wedge Z + (-1)^{kl+|X||Y|}Y \wedge [X, Z]_S; \tag{2.5}$$

and that for lower order graded tensors is defined by

(a) $[f, g]_S := 0$ if f and g are superfunctions;
(b) $[X, Y]_S := [X, Y]$ (the usual commutator) if X and Y are supervector fields;
(c) $[f, X]_S := df(X) = (-1)^{|f||X|}X(f)$ if $f \in \mathcal{A}$ and $X \in \mathfrak{X}(\mathcal{A})$.

Proposition 2.2. *Let $\mathcal{M}$ be a graded manifold, and let $\{\cdot\,,\cdot\}$ be an even Poisson structure on $\mathcal{M}$. If Δ is the graded tensor of type $(2,0)$ associated to this structure, then $[\Delta,\Delta]_S=0$.*

Proof. It is enough to verify the result locally, hence, by (2.4),

$$[\Delta,\Delta]_S=\sum_{kl}[\partial_{q^k}\wedge X_{q^k},\partial_{q^l}\wedge X_{q^l}]_S-\sum_{k\epsilon}[\partial_{q^k}\wedge X_{q^k},\partial_{\theta^\epsilon}\wedge X_{\theta^\epsilon}]_S$$
$$-\sum_{l\gamma}[\partial_{\theta^\gamma}\wedge X_{\theta^\gamma},\partial_{q^l}\wedge X_{q^l}]_S+\sum_{\gamma\epsilon}[\partial_{\theta^\gamma}\wedge X_{\theta^\gamma},\partial_{\theta^\epsilon}\wedge X_{\theta^\epsilon}]_S.$$

To compute each one of these terms one uses (2.5) and the equivalent identity

$$[X\wedge Y,Z]_S=X\wedge[Y,Z]_S+(-1)^{lr+|Y||Z|}[X,Z]_S\wedge Y.$$

For instance,

$$\begin{aligned}[\partial_{q^k}\wedge X_{q^k},\partial_{\theta^\epsilon}\wedge X_{\theta^\epsilon}]_S&=[\partial_{q^k}\wedge X_{q^k},\partial_{\theta^\epsilon}]_S\wedge X_{\theta^\epsilon}+\partial_{\theta^\epsilon}\wedge[\partial_{q^k}\wedge X_{q^k},X_{\theta^\epsilon}]_S\\&=\partial_{q^k}\wedge[X_{q^k},\partial_{\theta^\epsilon}]_S\wedge X_{\theta^\epsilon}-[\partial_{q^k},\partial_{\theta^\epsilon}]_S\wedge X_{q^k}\wedge X_{\theta^\epsilon}\\&\quad+\partial_{\theta^\epsilon}\wedge\partial_{q^k}\wedge[X_{q^k},X_{\theta^\epsilon}]_S-\partial_{\theta^\epsilon}\wedge[\partial_{q^k},X_{\theta^\epsilon}]_S\wedge X_{q^k}.\end{aligned}$$

To compute the non-vanishing terms of these expressions, notice that if the local expression of a supervector field is

$$X=\sum_{i=1}^m X^i\partial_{q^i}+\sum_{\alpha=1}^n\chi^\alpha\partial_{\theta^\alpha},$$

then

$$[X,\partial_{q^k}]=-\sum_{i=1}^m\frac{\partial X^i}{\partial q^k}\partial_{q^i}-\sum_{\alpha=1}^n\frac{\partial\chi^\alpha}{\partial q^k}\partial_{\theta^\alpha},$$
$$[X,\partial_{\theta^\epsilon}]=-(-1)^{|X|}\left(\sum_{i=1}^m\frac{\partial X^i}{\partial\theta^\epsilon}\partial_{q^i}+\sum_{\alpha=1}^n\frac{\partial\chi^\alpha}{\partial\theta^\epsilon}\partial_{\theta^\alpha}\right);$$

and since $X_{q^k}=\sum_j\Delta^{kj}\partial_{q^j}+\sum_\beta M^{k\beta}\partial_{\theta^\beta}$ and $X_{\theta^\epsilon}=\sum_j\mathcal{M}^{\epsilon j}\partial_{q^j}+\sum_\alpha S^{\epsilon\alpha}\partial_{\theta^\alpha}$, we get, using (2.3), that

$$\begin{aligned}&\sum_{k\epsilon}\partial_{q^k}\wedge[X_{q^k},\partial_{\theta^\epsilon}]_S\wedge X_{\theta^\epsilon}\\&=-\sum_{kij\epsilon}\partial_{q^k}\wedge\frac{\partial\Delta^{ki}}{\partial\theta^\epsilon}\partial_{q^i}\wedge M^{\epsilon j}\partial_{q^j}-\sum_{ki\beta\epsilon}\partial_{q^k}\wedge\frac{\partial\Delta^{ki}}{\partial\theta^\epsilon}\partial_{q^i}\wedge S^{\epsilon\beta}\partial_{\theta^\beta}\\&\quad-\sum_{k\alpha j\epsilon}\partial_{q^k}\wedge\frac{\partial M^{k\alpha}}{\partial\theta^\epsilon}\partial_{\theta^\alpha}\wedge M^{\epsilon j}\partial_{q^j}-\sum_{k\alpha\beta\epsilon}\partial_{q^k}\wedge\frac{\partial M^{k\alpha}}{\partial\theta^\epsilon}\partial_{\theta^\alpha}\wedge S^{\epsilon\beta}\partial_{\theta^\beta}\\&=\sum_{kij\epsilon}M^{j\epsilon}\frac{\partial\Delta^{ki}}{\partial\theta^\epsilon}\partial_{q^k}\wedge\partial_{q^i}\wedge\partial_{q^j}-\sum_{ki\beta\epsilon}S^{\beta\epsilon}\frac{\partial\Delta^{ki}}{\partial\theta^\epsilon}\partial_{q^k}\wedge\partial_{q^i}\wedge\partial_{\theta^\beta}\\&\quad-\sum_{k\alpha j\epsilon}M^{j\epsilon}\frac{\partial M^{k\alpha}}{\partial\theta^\epsilon}\partial_{q^k}\wedge\partial_{\theta^\alpha}\wedge\partial_{q^j}-\sum_{k\alpha\beta\epsilon}S^{\beta\epsilon}\frac{\partial M^{k\alpha}}{\partial\theta^\epsilon}\partial_{q^k}\wedge\partial_{\theta^\alpha}\wedge\partial_{\theta^\beta}.\end{aligned}\tag{2.6}$$

On the other hand, notice that

$$\{q^j,\{q^k,q^i\}\}=\{q^j,\Delta^{ki}\}=X_{q^j}(\Delta^{ki})=\sum_l\Delta^{jl}\frac{\partial\Delta^{ki}}{\partial q^l}+\sum_\epsilon M^{j\epsilon}\frac{\partial\Delta^{ki}}{\partial\theta^\epsilon}.\tag{2.7}$$

Thus, all the superfunctions that appear as coefficients in (2.6), and the like terms, are precisely terms that correspond to $\{q^j, \{q^k, q^i\}\}$, or similar expressions, hence to conclude it is enough to make sure that one has the right signs to be able to use the Jacobi identity in all possible combinations of the supercoordinates. $\square$

Remark 2.1. From the proof of Proposition 2.2, it is clear that, as in the non-graded case, the condition $[\Delta, \Delta]_S = 0$, is equivalent to the fact that the bracket giving rise to Δ satisfies the graded Jacobi identity.

Remark 2.2. On the other hand, it is not possible to describe odd Poisson structures in these terms. Even though the graded tensor Δ has some kind of "symmetry" (see 2.2), the set of all graded tensor with this property is not an algebra, since the equivalent to the antisymmetrization operator does not have similar properties as the antisymmetrization operator, and does not lead to a well defined product. In other words, we do not have a natural and reasonable algebra where to start the construction, so we cannot even define the Schouten bracket in the odd case.

It should be clear now that even Poisson structures lead to a geometry quite similar to non-graded geometry, whereas the odd ones present some new features. Nevertheless, when $\dim \mathcal{M} = (2m, 2m)$, it is possible to consider the case when one has a structure of each type at the same time, a situation that have been explored by Soroka in [24] and by Khudaverdian and Nersessian in [12–14]

3. Recursion operators

3.1. The graded-Nijenhuis tensor

In spite of the fact that even and odd graded Poisson manifolds do not have the same properties, one can develop the theory of Hamiltonian and locally Hamiltonian dynamical systems on graded manifolds in complete analogy with the classical theory, regardless of the parity of the graded sympletic form in consideration. Moreover, the relation between constants of motion and the symmetries of such systems, the so called supersymmetries, can be carried out, in full analogy with the classical case, using basically the same kind of proofs [8]. Nevertheless, another difference between the even and odd case is related to the Nijenhius tensor, and this have some implications with supersymmetries.

To define the graded-Nijenhuis tensor we use the notion of derivation of graded forms introduced by Kostant in [15]. Thus, a derivation of bidegree $(r, j) \in \mathbb{Z} \oplus \mathbb{Z}_2$ is a map

$$D\colon \Omega(\mathfrak{U}, \mathcal{A}) \to \Omega(\mathfrak{U}, \mathcal{A}),$$

for each open subset $\mathfrak{U}$ of M such that:

(a) $D\big(\Omega^k(\mathfrak{U}, \mathcal{A})_i\big) \subset \Omega^{k+r}(\mathfrak{U}, \mathcal{A})_{i+j} \quad \forall (k, i) \in \mathbb{Z} \oplus \mathbb{Z}_2$.

(b) $D(\omega + \nu) = D(\omega) + D(\nu)$.

(c) $D(\omega \wedge \nu) = D(\omega) \wedge \nu + (-1)^{kr+ij} \omega \wedge D(\nu)$, if $|\omega| = (k, i)$.

(d) $D(\lambda \mathbb{1}) = 0$, for each $\lambda \in \mathbb{R}$ (Here $\mathbb{1}$ denotes the identity of the superalgebra $\mathcal{A}(U)$).

Extending the standard arguments to graded geometry, one can check that these derivations are local operators, and therefore they are determined by their actions on 0-forms and exact 1-forms [5].

If $A\colon \mathfrak{X}(\mathcal{M}) \times \cdots \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ is a graded tensor of type $(r, 1)$, we denote by i_A the derivation of bidegree $(r-1, |A|)$ that vanish on each superfunction $f \in \mathcal{A}$, and that on exact graded 1-forms is given by

$$\big(i_A df\big)(X_1, \ldots, X_r) = (-1)^{|A||f|} df\big(A(X_1, \ldots, X_r)\big) \qquad X_i \in \mathfrak{X}(\mathcal{M}). \tag{3.1}$$

When T is a graded tensor of type $(1, 1)$, which is the case we are primarily interested in, one can prove, by induction on the order of the forms, that if $\omega \in \Omega^k(\mathcal{U}, \mathcal{A})$, then

$$i_T\omega(X_1, \ldots, X_k) = (-1)^{|T||\omega|} \sum_{\sigma \in S_k} (-1)^{|X_\sigma|} \omega(TX_{\sigma(1)}, \ldots, X_{\sigma(k)}), \tag{3.2}$$

where S_k denotes the group of permutations of k elements, $|X_\sigma| := \sum_{(i,j)\in\Delta_\sigma} 1 + |X_i||X_j|$, and $\Delta_\sigma = \{\,(i, j) : i < j \text{ and } \sigma(i) > \sigma(j)\,\}$.

The graded commutator of two derivations D_1 and D_2 of bidegrees (r_1, j_1) and (r_2, j_2), respectively, given by

$$[D_1, D_2] := D_1 D_2 - (-1)^{r_1 r_2 + j_1 j_2} D_2 D_1, \tag{3.3}$$

is again a derivation of bidegree $(r_1 + r_2, j_1 + j_2)$. Hence, we can also associate to the graded tensor A the derivation of bidegree $(r, |A|)$ defined by

$$d_A := [i_A, d] = i_A\, d + (-1)^r d\, i_A, \tag{3.4}$$

where d is the graded exterior derivative.

In non-graded geometry the Nijenhuis tensor associated to a tensor T of type $(1, 1)$ is usually defined as the tensor N_T of type $(1, 2)$ such that

$$d_{N_T} = [d_T, d_T].$$

If now T is a graded tensor of type $(1, 1)$ on $\mathcal{M}$, it is only natural to define the graded Nijenhuis tensor in the same way but using (3.3). Thus, we define N_T^G by the formula

$$d_{N_T^G} = [d_T, d_T] = d_T \circ d_T + (-1)^{|T|^2} d_T \circ d_T,$$

which does define a graded tensor of type $(2, 1)$. Nevertheless, when T is odd, $N_T^G \equiv 0$, and therefore the graded Nijenhuis tensor is only meaningful when $|T| = 0$. Moreover, from a long but simple computation using (3.2) and (3.4) one has

$$\begin{aligned} d_T \circ d_T f(X, Y) = (-1)^{|f|(|X|+|Y|)} \big\{ &(-1)^{|T|^2} T^2[X, Y](f) - (-1)^{|T|^2} T[TX, Y](f) \\ &+ (-1)^{|X||T|} \big(TX(TY(f)) - (-1)^{|X||Y| + |T||X| + |T||Y|} TY(TX(f))\big) \\ &- (-1)^{|T|^2 + |T||X|} T[X, TY](f) \big\}, \end{aligned}$$

for every superfunction $f \in \mathcal{A}$. Thus, when $|T| = 0$, using (3.1) and $d_{N_T^G} f = i_{N_T^G} df$, we obtain the usual expresion:

$$N_T^G(X, Y) = T^2[X, Y] + [TX, TY] - T[TX, Y] - T[X, TY] \qquad X, Y \in \mathfrak{X}(\mathcal{A}). \tag{3.5}$$

3.2. Recursion operators

In spite of the above, when the Nijenhuis tensor is meaningful it plays the same role as in the classical theory. Let Γ be a graded vector field in $\mathfrak{X}(\mathcal{A})$ and let $\{X_i, X_\alpha : i = 1, \dots, m, \alpha = 1, \dots, n\}$ be a local basis of $\mathfrak{X}\big(\mathcal{A}(U)\big)$. From now on, T will denote a Γ-invariant graded tensor of type $(1, 1)$ of $\mathcal{M}$ (i.e., $\mathcal{L}_\Gamma T = 0$), and

$$A = \begin{pmatrix} A_{ij} & A_{i\beta} \\ A_{\alpha j} & A_{\alpha\beta} \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} B_{ij} & B_{i\beta} \\ B_{\alpha j} & B_{\alpha\beta} \end{pmatrix},$$

will denote the matrix representation of T and $\mathcal{L}_\Gamma$ respectively, associated to the basis $\{X_i, X_\alpha\}$; in other words:

$$\begin{aligned} TX_j &= \sum_k X_k A_{kj} + \sum_\gamma X_\gamma A_{\gamma j}, \; TX_\beta = \sum_k X_k A_{k\beta} + \sum_\gamma X_\gamma A_{\gamma\beta}, \\ \mathcal{L}_\Gamma X_k &= \sum_k X_k B_{kj} + \sum_\gamma X_\gamma B_{\gamma j}, \; \mathcal{L}_\Gamma X_\beta = \sum_k X_k B_{k\beta} + \sum_\gamma X_\gamma B_{\gamma\beta}, \end{aligned} \tag{3.6}$$

where $A_{ij}, \dots, B_{\alpha\beta}$ are superfunctions on $\mathcal{A}(U)$.

Proposition 3.1. *If T is a Γ-invariant graded tensor of type $(1, 1)$ on $\mathcal{M}$, then* $\operatorname{str}(T)$ *is a constant of the motion generated by* Γ.

Proof. Since $\mathcal{L}_\Gamma T = 0$ then

$$\begin{aligned} 0 &= \big(\mathcal{L}_\Gamma T\big)(X_j) = \mathcal{L}_\Gamma\big(T(X_j)\big) - (-1)^{|\Gamma||T|} T(\mathcal{L}_\Gamma X_j) \\ &= \sum_i X_i \cdot \big\{[B, A]_{ij} + \Gamma(A_{ij})\big\} + \sum_\alpha X_\alpha \cdot \big\{[B, A]_{\alpha j} + (-1)^{|\Gamma|}\Gamma(A_{\alpha j})\big\}; \end{aligned}$$

and since the X_i and X_α are linearly independent we conclude that

$$[B, A]_{ij} = -\Gamma(A_{ij}) \qquad \text{and} \qquad [B, A]_{\alpha j} = -(-1)^{|\Gamma|}\Gamma(A_{\alpha j}).$$

Similarly, one can prove that $[B, A]_{i\beta} = -\Gamma(A_{i\beta})$ and $[B, A]_{\alpha\beta} = -(-1)^{|\Gamma|}\Gamma(A_{\alpha\beta})$.

On the other hand, since $\operatorname{str}(AB) = (-1)^{|B||A|}\operatorname{str}(BA)$, then $\operatorname{str}([A, B]) = 0$. Thus, if $\Gamma(A)$ is the graded matrix

$$\Gamma(A) := -\begin{pmatrix} \Gamma(A_{ij}) & \Gamma(A_{i\beta}) \\ (-1)^{|\Gamma|}\Gamma(A_{\alpha j}) & (-1)^{|\Gamma|}\Gamma(A_{\alpha\beta}) \end{pmatrix}, \tag{3.7}$$

we have

$$0 = \operatorname{str}(\Gamma(A)) = -\Gamma(A_{ii}) + (-1)^{|\Gamma(A)|+|\Gamma|}\Gamma(A_{\alpha\alpha}) = \Gamma\big(\operatorname{str}(A)\big);$$

in other words, $\operatorname{str}(T) = \operatorname{str}(A)$ is a constant of the motion. $\square$

Actually, not only $\operatorname{str}(T)$ is a constant of motion, but also $\operatorname{str}(T^k)$ is a constant of motion for each $k \in \mathbb{N}$, a fact that follows immediately from the next proposition.

Proposition 3.2. *If T is a Γ-invariant graded tensor field of type* $(1, 1)$, *then*

$$[B, A^p] = \Gamma(A^p),$$

where the matrix $\Gamma(A^p)$ *is defined as in* (3.7).

Proof. First of all, from a simple computation that involves nothing but good bookkeeping, one has

$$[B, A^{p+1}] = [B, A^p]A + (-1)^{|\Gamma||A^p|}A^p[B, A]. \tag{3.8}$$

The proposition will now be proved by induction: we know the relation holds for $p = 1$, and we assume it to be true for some interger p. Using that Γ is a graded derivation and some bookkeeping one establishes that

$$\Gamma(A^{p+1}) = \Gamma(A^p)A + (-1)^{|\Gamma||A^p|}A^p\Gamma(A).$$

The result then follows from the induction hypothesis and (3.8). $\square$

Definition 3.1. An operator $T: \mathfrak{X}(\mathcal{A}) \to \mathfrak{X}(\mathcal{A})$ is said to be a recursion operator for Γ if whenever $X \in \mathfrak{X}(\mathcal{A})$ is an infinitesimal supersymmetry of Γ (i.e., $\mathcal{L}_\Gamma X = 0$) also $T(X)$ is an infinitesimal supersymmetry of Γ.

It is clear that a Γ-invariant graded tensor field of type $(1, 1)$ is a recursion operator. Actually one has a full sequence of infinitesimal symmetries: $X, T(X), T^2(X), \ldots$. Moreover, these symmetries generate a supercommutative superalgebra.

Proposition 3.1. *If T is an even Γ-invariant graded tensor of type* $(1, 1)$, *whose graded-Nijenhuis tensor vanishes, and X is an infinitesimal supersymmetry of Γ, then the superalgebra generated by the T^pX's is supercommutative.*

Proof. The proof involves several inductions: first of all, we prove that

$$[X, T^kX] = 0 \qquad \forall k \in \mathbb{N}. \tag{3.9}$$

Formula (3.9) is obvious for $k = 0$, and if it holds for some p, then

$$[X, T^{p+1}X] = \mathcal{L}_X(T^{p+1}X) = (\mathcal{L}_XT)(T^pX) + (-1)^{|X||T|}T\big(\mathcal{L}_XT^p(X)\big) = 0,$$

which completes the first induction.

Now we shall prove, also by induction, that

$$[T^kX, T^pX] = 0 \qquad \forall k, p \in \mathbb{N}.$$

Clearly $[X, TX] = 0 = [TX, TX]$. Let us suppose that for some fixed p the equation $[T^kX, T^pX] = 0$ holds whenever $k \leqslant p$. To complete the induction we have to prove that $[T^kX, T^{p+1}X] = 0$, if $k \leqslant p + 1$. Once again we do this using induction. We know that $[X, T^{p+1}X] = 0$; and if for some $l < p$

$$[T^lX, T^{p+1}X] = 0,$$

then using both induction hypotheses and (3.5),

$$\begin{aligned} N_T^G(T^l X, T^p X) &= T^2[T^l X, T^p X] + [T^{l+1} X, T^{p+1} X] \\ &\quad - T[T^{l+1} X, T^p X] - T[T^l X, T^{p+1} X] \\ &= [T^{l+1} X, T^{p+1} X], \end{aligned}$$

and since $N_T^G \equiv 0$ the proposition is proved. □

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